

STABLE INTERACTIONS VIA PROPER DEFORMATIONS

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ABSTRACT. A new method is proposed for switching on interactions that are compatible with global symmetries and conservation laws of the original free theory. The method is applied to the control of stability in Lagrangian and non-Lagrangian theories with higher derivatives. By way of illustration, a wide class of stable interactions is constructed for the Pais-Uhlenbeck oscillator.

1. INTRODUCTION

The inclusion of consistent interactions is a notorious problem in various areas of field theory. The problem has several aspects related to the notion of consistency. In gauge theories, for instance, consistency is usually understood as the requirement that the theory still has the same number of gauge symmetries as it has had before inclusion of interaction. This requirement is necessary (but not always sufficient) to ensure that the free and interacting models possess the same number of physical degrees of freedom. Nowadays, the BRST theory provides the most powerful approach to the control of gauge symmetries upon switching on interaction [1, 2, 3, 4]. A complete control over physical degrees of freedom is achieved in the involutive form of dynamics. Using the concept of involution, a covariant perturbative procedure for inclusion of interaction was proposed in [5]. Apart from gauge symmetries, the procedure accounts for hidden integrability conditions (constraints) making no distinction between Lagrangian and non-Lagrangian theories.

The stability of nonlinear dynamics is another crucial property of interaction. Being understood as the boundedness of solutions to the classical equations of motion, it provides a sufficient condition for the existence of a stable quantum theory with a well-defined vacuum state. This relationship between the classical and quantum stability is almost obvious in theories without higher derivatives. Once the energy is bounded¹, the theory is stable because each classical trajectory lies on a bounded isoenergetic surface in the phase space and the quantum vacuum can be defined as

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¹By a bounded function we mean a function satisfying the two requirements: (i) the function is bounded from below *and* (ii) the level surfaces of the function are bounded in the ambient space.

the state with the lowest energy. The sufficient stability condition, however, becomes an issue in the higher-derivative theories, where the canonical energy is usually unbounded even in the linear approximation. For an introductory discussion of the stability problem in higher-derivative theories we refer the reader to [6]. The simplest example of higher-derivative dynamics is provided by the Pais-Uhlenbeck oscillator. The stability of this model at the free and interacting levels has been the subject of numerous studies, see e.g. [7, 8, 9, 10, 11] and references therein.

In the recent papers [12, 13], a new non-perturbative approach to the stability of higher-derivative systems has been proposed. The key ingredient of the approach is the concept of Lagrange structure [14, 15, 16]. The role of the Lagrange structure is twofold. On the one hand, it makes possible a consistent quantization of a classical system even though the classical equations of motion are non-Lagrangian, on the other hand each Lagrange structure defines a specific correspondence between symmetries and conservation laws of the theory. The latter property can be viewed as an extension of the Noether theorem to the non-Lagrangian theories [17]. Once the classical equations of motion admit a bounded integral of motion, and the Lagrange structure that relates this integral to the time-translation symmetry, the theory can retain stability at the quantum level. The bounded integral of motion, being connected with the time translation, is naturally identified with a physical energy of the system, which may differ from the canonical energy. The surprising thing is that such integrals of motion and Lagrange structures exist almost for any free theory. Furthermore, their number increases (in some precise sense) with increasing the order of equations of motion. Upon inclusion of interaction the equations of motion and the Lagrange structure should be deformed in such a way as to keep the bounded conservation law connected to the time translation. In the case that the operator of free higher-order equations admits factorization into the product of coprime, lower-order operators, particular solutions to this deformation problem were proposed in [12, 13, 18].

In the present work, we construct a new class of stable nonlinear theories with higher derivatives, where the interaction is introduced by the proper deformation method proposed in [17]. Contrary to the method of [12], the proper deformation of classical equations does not change the Lagrange structure, deforming only the characteristic of the conservation law. Under certain conditions the deformed equations describe a stable dynamical system, at least in some vicinity of classical vacuum. It is significant that the resulting non-linear theory is always Hamiltonian, even though the interaction vertices introduced by proper deformation are generally non-Lagrangian.

The paper is organized as follows. In Sec. 2, we recall the definition of the Lagrange structure and the formulation of the generalized Noether theorem for not necessarily Lagrangian theories.

The main results of the paper are contained in Sec. 3. Here, after explaining the notion of a proper deformation, a set of conditions is specified whereby a given symmetry of equations of motion remains unchanged, while the corresponding integral of motion gets linear and quadratic corrections in deformation. In Sec. 4, we apply the proper deformation technique to the translation-invariant theories and construct a class of stable non-linear systems with higher derivatives. The general method is then illustrated by the example of Pais-Uhlenbeck oscillator. In the concluding Sec. 5 we summarize the results.

2. CONSERVATION LAWS, SYMMETRIES AND LAGRANGE STRUCTURES

We start with a brief review of the concept of a Lagrange structure and its relation to the symmetries and conservation laws. To simplify formulas below we use the condensed index notation [19]. According to this notation, the set of fields $\{\varphi^i\}$ is labeled by the index i that includes all discrete indices as well as local coordinates x^μ on a d -dimensional spacetime manifold M . As usual, summation over repeated condensed indices implies integration over the spacetime and the partial derivatives $\partial_i = \partial/\partial\varphi^i$ are understood as variational ones. The equation $S \doteq 0$ means that a local functional $S[\varphi]$ is given by an integral over M of an exact d -form, $S = \int_M dJ$; and hence, $\partial_i S = 0$.

The dynamics of fields are governed by a set of partial differential equations

$$(1) \quad T_a(\varphi) = 0.$$

Since we do not assume the equations of motion to come from the least action principle, the (discrete part of) indices a and i may run over different sets. For Lagrangian theories $a = i$ and $T_i = \partial_i S$, with S being a classical action. The set of all solutions to (1) is called the *mass shell* or simply *shell*.

By definition, a *conservation law* J is given by an on-shell closed differential form on M of degree $d - 1$. Under the standard regularity conditions on the equations of motion (1) this is equivalent to

$$(2) \quad Q^a T_a = \int_M dJ \doteq 0.$$

The set of coefficients $Q = \{Q^a\}$ defining the left hand side is known as the *characteristic* of the conservation law J . Like T 's, the components of the form J and the characteristic Q are supposed to be smooth functions of fields and their space-time derivatives. It is known that modulo some

trivialities there is a one-to-one correspondence between the conservation laws and characteristics [17, 20].

A variational vector field $X = X^i \partial_i$ is called a *symmetry* of the equations of motion (1) if it preserves the mass shell, i.e.,

$$(3) \quad XT_a = U_a^b T_b$$

for some structure functions U_a^b . Two symmetries X_1 and X_2 are considered as equivalent if they coincide on shell, i.e.,

$$X_1 - X_2 = T_a K^a, \quad (U_1)_a^b - (U_2)_a^b = K^b T_a$$

for some set of variational vector fields $K^a = K^{ai} \partial_i$. It should be emphasized that for non-Lagrangian theories the symmetries and conservation laws are not related by the Noether theorem anymore.

A set of variational vector fields $V_a = V_a^i \partial_i$ is said to define a *Lagrange structure* for the equations of motion (1) if the following compatibility condition is satisfied:

$$(4) \quad V_a T_b - V_b T_a = C_{ab}^d T_d$$

for some structure functions $C_{ab}^d = -C_{ba}^d$. The distribution $V = \{V_a\}$ is called the *Lagrange anchor*.

In this paper, we are mostly interested in the so-called *strongly integrable Lagrange structures*. These satisfy the following additional conditions:

$$(5) \quad [V_a, V_b] = C_{ab}^d V_d, \quad V_a C_{bd}^f + C_{ab}^g C_{dg}^f + \text{cycle}(a, b, d) = 0.$$

The first relation just says that the anchor distribution V is integrable. For linearly independent V 's the second relation is then a mere consequence of the Jacobi identity for the Lie bracket of vector fields. From the geometrical viewpoint, Rels. (5) define a Lie algebroid with anchor $V = \{V_a\}$ and Rel. (4) can be regarded as the closedness condition for the Lie algebroid one-form $T = \{T_a\}$. For a quick introduction to the theory of Lie algebroids we refer the reader to [21].

Notice that each Lagrangian theory admits the *canonical Lagrange anchor* $\{V_i = \partial_i\}$ associated with the tangent Lie algebroid. In that case, defining relation (4) is automatically satisfied due to the commutativity of variational derivatives,

$$\partial_i T_j - \partial_j T_i = \partial_i \partial_j S - \partial_j \partial_i S = 0, \quad C_{ij}^k = 0.$$

It should be noted that the existence of a Lagrange structure compatible with a given set of equations appears to be much less restrictive condition for the dynamics than the existence of an action. Many examples of non-Lagrangian equations together with their Lagrange structures can be found in [14, 22, 23, 24]. Let us stress that the choice of a compatible Lagrange structure is not unique, and even Lagrangian equations of motion may have a variety of different (and hence, non-canonical) Lagrange structures. A particular example of such a situation will be considered in Sec. 4.

In [17], it was shown that each Lagrange anchor (be it integrable or not) establishes a relationship between the conservation laws and symmetries. Explicitly,

$$(6) \quad X = Q^a V_a ,$$

where Q is the characteristic of a conservation law (2). Using (4) one can easily see that

$$XT_a = (C_{ba}^d Q^b - V_a Q^d) T_d .$$

The symmetries of the form (6) are called the *characteristic symmetries*.

We see that the correspondence between symmetries and conservation laws is not given from the outset, but depends upon the choice of a Lagrange anchor. The classical Noether's theorem exploits the canonical Lagrange anchor for Lagrangian equations of motion. In the general case a dynamical system may admit several Lagrange anchors leading to different relations between symmetries and conservation laws. This means, in particular, that one and the same symmetry may come from different conservation laws. For the translation-invariant equations of motion, this allows one to construct a multi-parameter family of Hamiltonians associated with various Lagrange anchors. In some cases, this family may contain a positive-definite Hamiltonian even in the presence of higher derivatives. Some examples of this kind can be found in [12, 13, 25, 26].

3. PROPER DEFORMATIONS AND CONSERVATION LAWS

The second key ingredient of our construction, called the *proper deformation*, was introduced in [17]. This is defined as follows. Let we have given two sets of equations of motion,

$$(7) \quad T_a(\varphi) = 0 \quad \text{and} \quad T'_a(\varphi) = 0 ,$$

for one and the same collection of fields $\{\varphi^i\}$ and let V be a strongly integrable Lagrange anchor for T 's. We say that the second set of equations is obtained by a proper deformation of the first

one if there exists a local functional \mathcal{S} , called the *generator of the proper deformation*, such that

$$(8) \quad T'_a = T_a + V_a \mathcal{S}.$$

By making use of Rels. (5) one can easily see that the Lagrange anchor V is also compatible with the deformed equations, so that both theories (7) share the same Lagrange structure. One can also regard Eq. (8) as an equivalence relation on the space of all equations of motion compatible with a given Lagrange anchor V . In general, the corresponding equivalence classes may be rather wide. For example, any two Lagrangian theories are related by a proper deformation w.r.t. the canonical Lagrange anchor:

$$T'_i = T_i + \partial_i \mathcal{S}, \quad T_i = \partial_i \mathcal{S}, \quad T'_i = \partial_i \mathcal{S}', \quad \mathcal{S} = \mathcal{S}' - \mathcal{S}.$$

Suppose now that the first theory in (7) has a symmetry generated by the variational field X which leaves invariant the generator of proper deformation and the Lagrange anchor in the sense that

$$(9) \quad X\mathcal{S} \doteq 0, \quad [X, V_a] = U_a^b V_b,$$

with U 's being given by (3). Then, using definition (4), one can find

$$XT'_a = U_a^b T'_b.$$

The last relation tells us that X is a symmetry of the deformed equations, too. Furthermore, if X is a characteristic symmetry (6) of the first theory, then, under certain conditions to be specified below, it remains so in the deformed theory.

As was mentioned in the previous section, the generators of symmetry are defined only modulo the equations of motion. Given a characteristic symmetry, the general element of its equivalence class reads

$$(10) \quad X = Q^a V_a + T_a K^a,$$

with $K^a = K^{ai} \partial_i$ being some set of variational vector fields. Let us further assume that K 's satisfy the relation

$$(11) \quad K^{ai} V_a^j + K^{aj} V_a^i = 0.$$

(More geometrically, the last condition can be written as $(V_a \mathcal{S})(K^a \mathcal{S}) \doteq 0$ for all local functionals \mathcal{S} .) If J is a conservation law of the first theory with characteristic Q , then the deformed theory

(8) possesses the conservation law $J' = J + J_1 + J_2$, where the $(d-1)$ -forms J_1 and J_2 are defined by the relations

$$(12) \quad \int_M dJ_1 = Q^a(V_a \mathcal{S}) - T_a(K^a \mathcal{S}), \quad \int_M dJ_2 = (K^a \mathcal{S})(V_a \mathcal{S}).$$

The characteristic Q' of J' is given by

$$(13) \quad Q'^a = Q^a + K^a \mathcal{S}, \quad \int dJ' = Q'^a T'_a.$$

It is easy to see that the symmetry associated with this characteristic is equivalent to X ,

$$Q'^a V_a = X - T'_a K^a.$$

We thus conclude that, under condition (11), each integrable Lagrange structure allows one to deform equations of motion together with their conservation laws; in so doing, the deformed and undeformed conservation laws correspond to essentially the same symmetry transformation on the configuration space of fields. This observation will be used to the control of stability in the next section.

4. APPLICATION TO THE STABILITY OF INTERACTIONS

In this section, we consider mechanical systems whose dynamics are governed by ordinary differential equations (not necessarily Lagrangian). We say that a system is *classically stable* if each its trajectory is bounded in the phase space. In particular, this ensures the boundedness of motion in the configuration space. It may happen that a classical system becomes stable when restricted to some invariant domain in the phase space. Such a domain is usually referred to as a stability island. Generally, it is not easy to decide whether a given set of equations defines a stable system or system with stability islands. In most cases the classical stability is provided by an integral of motion whose level surfaces are bounded in the phase space. If in addition the values of the integral are bounded from below we call it bounded. For Lagrangian theories without higher derivatives the role of such an integral is often played by the canonical energy. Upon canonical quantization the energy becomes a Hermitian operator with spectrum bounded from below. This allows one to define the ground state as the state with the smallest possible energy. It is the existence of a ground (or vacuum) state which is usually understood by the quantum stability.

Unfortunately, the energy argument above can not be applied directly to the higher-derivative systems as the canonical Ostrogradsky's energy of such systems is known to be unbounded, at

least for regular Lagrangians. This does not necessarily mean that the system has no *other* integrals of motion, some of which may happen to be bounded as opposed to the canonical energy. Actually any bounded integral ensures the classical stability and one can try to interpret it as physical energy. To justify such an interpretation one only needs to find a Lagrange structure which would relate this integral to the time translation. On quantizing the theory by means of the Lagrange structure, this bounded integral of motion should be identified with the quantum Hamiltonian. By the correspondence principle one might expect the spectrum of this Hamiltonian to be bounded from below.

Finding of a bounded integral of motion for a given higher-derivative system is quite a difficult problem in general. The exception is provided by the linear higher-derivative systems, where one can usually find a plenty of integrals of motion with the desired property as well as Lagrangian structures linking them to the time-translation symmetry. The proper deformation gives a simple method for constructing nonlinear theories with conserved quantities related to the time translation. Whenever a linear model admits a bounded integral of motion and Eqs. (9), (10), (11) are satisfied, the conservation law J' of the corresponding nonlinear theory is given by (2) and (12) with $X = -\dot{\varphi}^i \frac{\partial}{\partial \varphi^i}$. With a suitable choice of S , the function J' can be made bounded, so that the nonlinear theory remains stable.

Let us now illustrate this general approach by the example of the Pais-Uhlenbeck oscillator. The theory is described by the forth-order differential equation

$$(14) \quad T(x) = \gamma(\overset{(4)}{x} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2x) = 0, \quad \overset{(k)}{x} = \frac{d^k x}{dt^k}, \quad \gamma = \frac{1}{\omega_2^2 - \omega_1^2},$$

where $0 < \omega_1 < \omega_2$ are the frequencies and $x(t)$ is a single dynamical variable. We exclude the case of equal frequencies as the corresponding motion is known to be unbounded due to the phenomenon of resonance.

In [12], the following two-parameter families of Lagrange anchors and characteristics were found:

$$(15) \quad V = -\gamma\left(\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{d^2}{dt^2} + \left(\frac{\omega_2^2}{\alpha} + \frac{\omega_1^2}{\beta}\right)\right), \quad Q = \gamma((\alpha + \beta)\ddot{x} + (\alpha\omega_2^2 + \omega_1^2)\dot{x}).$$

For any nonzero constants α and β they result to the time-translation symmetry

$$(16) \quad X \equiv -\dot{x} = V(Q) + K(T), \quad K = -\gamma \frac{(\alpha + \beta)^2}{\alpha\beta} \frac{d}{dt},$$

Hereafter, to be more explicit we unfold our condensed notation treating V_a^i and K^{ai} as integral kernels of differential operators acting on the test function $\zeta(t)$,

$$(V\zeta)(t) = \int dt' V(t, t') \zeta(t'), \quad (K\zeta)(t) = \int dt' K(t, t') \zeta(t'),$$

where $t = i$ and $t' = a$ are pure continuous indices. The integral of motion J corresponding to the characteristic (15) reads

$$(17) \quad J = \frac{\gamma^2}{2} (\alpha(\ddot{x} + \omega_2^2 \dot{x})^2 + \beta(\ddot{x} + \omega_1^2 \dot{x})^2 + \alpha\omega_1^2(\ddot{x} + \omega_2^2 \dot{x})^2 + \beta\omega_2^2(\ddot{x} + \omega_1^2 \dot{x})^2).$$

As is seen, the quadratic function J is positive definite provided that $\alpha, \beta > 0$. Furthermore, the underlying quadratic form on the phase-space of variables $x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}$ is nondegenerate whenever $\omega_1 \neq \omega_2$, so that each level surface of J appears to be compact and the trajectories are bounded. Notice that the canonical Ostrogradsky's energy corresponds to $\alpha = -\beta$, in which case V reduces to the canonical Lagrange anchor for the Lagrangian equations of motion. Being intimately related with the time-translation, the conserved quantity J can be regarded as a physical energy of the Pais-Uhlenbeck oscillator. This can be made positive by a proper choice of the free parameters α and β .

Applying to (14) a proper deformation generated by a local functional²

$$(18) \quad \mathcal{S} = -\frac{1}{\gamma} \int dt U(x, \dot{x}),$$

we get

$$(19) \quad \gamma(\overset{(4)}{x} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2 x) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\ddot{U}_x + \left(\frac{\omega_2^2}{\alpha} + \frac{\omega_1^2}{\beta}\right)U_x = 0,$$

where U_x denotes the Euler-Lagrange derivative of \mathcal{S} . It is clear that the functional \mathcal{S} is invariant under the time translations.

The linear and quadratic corrections to the integral of motion (17) due to the deformation are given by

$$\begin{aligned} J_1 &= \frac{1}{\gamma} U - \frac{1}{\gamma} \dot{x} \frac{\partial U}{\partial \dot{x}} + \gamma \frac{(\alpha + \beta)^2}{\alpha\beta} (\ddot{x} \dot{U}_x - \overset{(4)}{x} U_x) + \gamma \left(\frac{\omega_1^2}{\alpha} + \frac{\omega_2^2}{\beta} \right) (\alpha + \beta) (\dot{x} \dot{U}_x - \ddot{x} U_x), \\ J_2 &= -\frac{1}{2} \frac{(\alpha + \beta)^3}{\alpha^2 \beta^2} (2\ddot{U}_x U_x - \dot{U}_x^2) - \frac{1}{2} \frac{(\alpha + \beta)^2}{\alpha\beta} \left(\frac{\omega_2^2}{\alpha} + \frac{\omega_1^2}{\beta} \right) U_x^2. \end{aligned}$$

²The consistency of interaction [5] implies that both the linear and nonlinear theories have the same number of degrees of freedom. This forces us to restrict to the ansatz (18) without higher derivatives.

Excluding the fourth derivative in this expressions by means of equation (19), we can write the deformed integral of motion in the form

$$(20) \quad J' = \frac{\gamma^2}{2} \left\{ \alpha \left(\ddot{x} + \omega_2^2 \dot{x} + \frac{\alpha + \beta}{\alpha\beta\gamma} \dot{U}_x \right)^2 + \beta \left(\ddot{x} + \omega_1^2 \dot{x} + \frac{\alpha + \beta}{\alpha\beta\gamma} \dot{U}_x \right)^2 + \right. \\ \left. + \alpha \omega_1^2 \left(\ddot{x} + \omega_2^2 x + \frac{\alpha + \beta}{\alpha\beta\gamma} U_x \right)^2 + \beta \omega_2^2 \left(\ddot{x} + \omega_1^2 x + \frac{\alpha + \beta}{\alpha\beta\gamma} U_x \right)^2 \right\} + \frac{1}{\gamma} U - \frac{1}{\gamma} \dot{x} \frac{\partial U}{\partial \dot{x}}.$$

In general, this expression is not positive definite due to the negative terms in the third line. However, if the interaction U is small enough the integral J may well be bounded from below at least in some neighborhood of zero in the phase-space. For example, if we set $\alpha = \beta > 0$, then J' reduces to

$$(21) \quad J' = \frac{\alpha\gamma^2}{2} \left\{ \left(\ddot{x} + \omega_2^2 \dot{x} + \frac{2}{\alpha\gamma} \dot{U}_x \right)^2 + \left(\ddot{x} + \omega_1^2 \dot{x} + \frac{2}{\alpha\gamma} \dot{U}_x \right)^2 + \right. \\ \left. + \omega_1^2 \left(\ddot{x} + \omega_2^2 x + \frac{2}{\alpha\gamma} U_x \right)^2 + \omega_2^2 \left(\ddot{x} + \omega_1^2 x + \frac{2}{\alpha\gamma} U_x \right)^2 \right\} + \frac{1}{\gamma} U - \frac{1}{\gamma} \dot{x} \frac{\partial U}{\partial \dot{x}}.$$

This expression is clearly positive on the whole phase space whenever

$$U - \dot{x} \frac{\partial U}{\partial \dot{x}} > 0.$$

The simplest possibility to satisfy this inequality is to take $U = U(x) \geq 0$.

Notice that the equation describing the nonlinear Pais-Uhlenbeck oscillator (19) is non-Lagrangian, so that the classical Noether's theorem is not applicable to it anymore and the existence of the conserved quantity J' is not a mere consequence of the translation-invariance of the theory. However, applying the techniques developed in [27, 28], it is still possible to bring the nonlinear equation (19) into the Hamiltonian form with the function J' playing the role of Hamiltonian on the phase space of $x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}$. We are going to present the corresponding Poisson bracket elsewhere.

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